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Ito's theorem and gauge field theory

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Abstract. Perturbation theory is used to examine the stochastic quantisation hypothesis for gauge field theory. The generating functional is found to be badly behaved. A simple modification to the hypothesis is suggested which leads to the Fadeev-Popov expression for the generating functional.

1. Introduction

Recently considerable attention has been paid to the similarities between Euclidean quantum field theory and the theory of functional stochastic differential equations. Both theories are characterised by their Green functions, i.e. averages of products of field variables. Parisi and Wu (1981) conjectured that it is possible to choose the form of the stochastic differential equation so that both theories produce the same Green functions. This leads to a new (the so-called stochastic) procedure for quantising fields.

Stochastic quantisation presents several practical and conceptual possibilities not found in the functional integral formulation. For example the computer simulation of stochastic differential equations offers a new numerical technique in quantum field theory. Parisi (1981, 1982) has argued that this technique has better convergence properties than the more commonly used Monte Carlo simulation. Further work on algorithms for the numerical simulation of stochastic differential equations has been carried out by Drummond *et al* (1983) and Thomas (1984).

The stochastic procedure can be used to quantise classical field theories which cannot be formulated in terms of a least action principle. Zwanziger (1981) made use of this fact to give a new covariant quantisation of the Yang-Mills gauge field. The subject of this paper is also gauge field theory but the analysis differs both technically and conceptually from Zwanziger's work.

In the stochastic approach to gauge theory a random field $A_\mu^a(x, \tau)$ (μ Lorentz index, a group index) is defined over spacetime coordinates x plus an additional timelike coordinate τ. The evolution of the field with τ is assumed to be governed by the Langevin equation

$$\mathrm{d}A_\mu^a(x, \tau) = -\frac{\delta S[A]}{\delta A_\mu^a(x, \tau)}\,\mathrm{d}\tau + \sqrt{2}\,\mathrm{d}W_\mu^a(x), \tag{1.1}$$

where $S[A]$ is the classical action of the Yang-Mills field continued into Euclidean space and W_μ^a is a white noise field. For gauge invariant functionals $G[A]$ of the gauge field Parisi and Wu postulated that

$$\lim_{\tau\to\infty} \langle G[A(x, \tau)]\rangle_{\text{noise}} = \langle G[A(x)]\rangle_{\text{FT}}. \tag{1.2}$$

The subscript 'noise' denotes the ensemble average over the white noise field and 'FT' the usual quantum field theory vacuum expectation value.

In order to verify equation (1.2) it is necessary to calculate ensemble averages of functionals of the gauge field. Zwanziger (1981) and Floratos and Iliopoulos (1983) evaluated these averages as functional integrals over a τ-dependent probability density $P[A(x, \tau)]$ obeying the Fokker–Planck equation. A more direct approach based on the Ito stochastic calculus† is adopted here.

For any non-anticipating functional $F[A]$ of the field $A_\mu^a(x, \tau)$ obeying equation (1.1) Ito's theorem states that

$$\frac{\partial}{\partial\tau}\langle F\rangle_{\text{noise}} = \left\langle \int \mathrm{d}y \left(-\frac{\delta S}{\delta A_\mu^a(y,\tau)}\frac{\delta F}{\delta A_\mu^a(y,\tau)} + \frac{\delta^2 F}{\delta A_\mu^a(y,\tau)\delta A_\mu^a(y,\tau)}\right)\right\rangle_{\text{noise}}. \tag{1.3}$$

In particular the generating functional

$$Z(\tau) = \left\langle \exp \int \mathrm{d}x\, J_\mu^a(x) A_\mu^a(x, \tau)\right\rangle_{\text{noise}}, \tag{1.4}$$

where J_μ^a is a classical source field, is found to obey

$$\frac{\partial Z(\tau)}{\partial\tau} = \int \mathrm{d}y \left[-J_\mu^a(y)\frac{\delta S}{\delta A_\mu^a}\left(\frac{\delta}{\delta J(y)}\right) + J_\mu^a(y) J_\mu^a(y)\right] Z(\tau). \tag{1.5}$$

Note that this equation only determines Z up to a normalisation constant which is fixed by the initial condition $Z(\tau = 0) = 1$ and the boundary condition $Z[J = 0] = 1$. In the next section perturbation theory is used to analyse the equation for the generating functional.

2. Perturbation theory for the generating functional

The gradient of the action can be written

$$\frac{\delta S}{\delta A_\mu^a} = \frac{\delta S^{(0)}}{\delta A_\mu^a} + g\frac{\delta S^{(1)}}{\delta A_\mu^a} + g^2\frac{\delta S^{(2)}}{\delta A_\mu^a}, \tag{2.1}$$

where g is the coupling constant. On the right-hand side the first term is linear in the gauge field, the second quadratic and the third cubic. Equation (1.5) for the generating functional is therefore a third-order differential equation. However if the coupling constant is small the equation can be regarded as first order with the higher derivative terms being small perturbations.

Writing

$$Z(\tau) = \sum_{n=0}^{\infty} g^n Z_n(\tau), \tag{2.2}$$

substituting into equation (1.5) and equating powers of g gives

$$\frac{\partial Z_n}{\partial\tau}(\tau) + \int \mathrm{d}y \left[J_\mu^a(y)\frac{\delta S^{(0)}}{\delta A_\mu^a}\left(\frac{\delta}{\delta J(y)}\right) - J_\mu^a(y) J_\mu^a(y)\right] Z_n(\tau) = V_n^{(1)} + V_n^{(2)} \tag{2.3}$$

† For an introduction to the Ito stochastic calculus see Schuss (1980).

where

$$V_n^{(N)} = \int \mathrm{d}y\, J_\mu^a(y) \frac{\delta S^{(N)}}{\delta A_\mu^a} \left(\frac{\delta}{\delta J(y)}\right) Z_{n-N} \qquad n \geq N$$

and zero otherwise. This is a system of linear first-order differential equations.

The solution is most easily achieved in momentum space. Writing the left-hand side of equation (2.3) in momentum space gives

$$\left[\frac{\partial}{\partial \tau} + \int \mathrm{d}p\, J_\mu^a(p)\left(p^2 P_{\mu\nu}(p) \frac{\delta}{\delta J_\mu^a(p)} - J_\mu^a(-p)\right)\right] Z_n(\tau) = V_n^{(1)} + V_n^{(2)}, \tag{2.4}$$

where

$$P_{\mu\nu}(p) = \delta_{\mu\nu} - p_\mu p_\nu / p^2. \tag{2.5}$$

Using the method of characteristics the partial differential equation (2.4) can be converted to an ordinary differential equation. Along the curve

$$\mathrm{d}J_\mu^a(p)/\mathrm{d}\tau = p^2 P_{\mu\nu}(p) J_\nu^a(p), \tag{2.6}$$

equation (2.4) can be written as

$$\frac{\mathrm{d}Z_n}{\mathrm{d}\tau} - \int \mathrm{d}p\, J_\mu^a(p) J_\mu^a(-p) Z_n = V_n^{(1)} + V_n^{(2)}. \tag{2.7}$$

In equation (2.7) J_μ^a is to be regarded as a function of τ (that given by equation (2.6)), i.e. by

$$J_\mu^a(p, \tau) = (P_{\mu\nu}(p) \exp(p^2\tau) + p_\mu p_\nu / p^2) J_\nu^a(p, 0). \tag{2.8}$$

The solution of the simple ordinary differential equation (2.7) is carried out by multiplying through by the integrating factor

$$I = \exp\left[-\int^\tau \mathrm{d}\tau\left(\int \mathrm{d}p\, J_\mu^a(p) J_\mu^a(-p)\right)\right]. \tag{2.9}$$

Using equation (2.8) the integrating factor is found to be

$$I = \exp\left(-\tfrac{1}{2}\int \mathrm{d}p\, J_\mu^a(p)(P_{\mu\nu}(p) + 2p_\mu p_\nu \tau / p^2) J_\nu^a(-p)\right) \tag{2.10}$$

and the free field generating functional to be

$$Z_0 = \exp\left(\tfrac{1}{2}\int \mathrm{d}p\, J_\mu^a(p)(P_{\mu\nu}(p)[1 - \exp(-2p^2\tau)] + 2p_\mu p_\nu \tau / p^2) J_\nu^a(-p)\right). \tag{2.11}$$

From equations (2.10) and (2.11) it is easily seen that the Z_n do not have a stationary large τ limit. This observation does not rule out Parisi and Wu's postulate, equation (1.2), for gauge invariant functionals. It does, however, make comparison with the functional integral approach more difficult. The easiest case to consider is the free field, where it can be seen from equation (2.11) that any functional that can be built out of

$$A_\mu^T(p, \tau) = p_{\mu\nu}(p) A_\nu(p, \tau) \tag{2.12}$$

obeys the stochastic quantisation hypothesis. For the interacting theory the ratio of successive Z_n is not necessarily small and it is difficult to decide if the series for Z is meaningful.

It is therefore interesting to consider modifications of the stochastic quantisation hypothesis such that the Z_n have a stationary large τ limit. The problem term in equations (2.10) and (2.11) is $p_\mu p_\nu \tau/p^2$. This term arises because the operator $P_{\mu\nu}(p)$ in equation (2.6) has a zero eigenmode. Two distinct ways of removing this problem term are possible: either equation (2.6) can be modified so that the operator $P_{\mu\nu}(p)$ is replaced by a new operator without the zero eigenmode or equation (2.7) can be altered so that there is no contribution from the zero eigenmode. The first possibility has been explored by Zwanziger (1981). In the next section the second possibility will be discussed and shown to have similarities with the Fadeev-Popov construction in the functional integral formulation of gauge field theory.

3. Projection operators and constraints

A modification of equation (2.7) is required which allows a stationary propagator to be defined. The modified equation

$$\frac{dZ_0}{d\tau} - \int dp\, J^a_\mu(-p) P_{\mu\nu}(p) J^a_\nu(p) Z_0 = 0 \tag{3.1}$$

satisfies this requirement because equation (2.6) can now be used to give

$$\frac{d}{d\tau}\left[Z_0 \exp\left(-\tfrac{1}{2}\int dp\, J^a_\mu(-p) J^a_\mu(p)/p^2\right)\right] = 0. \tag{3.2}$$

The solution of this equation with the initial condition $Z_0(\tau=0)=1$ is found, using equation (2.8) and its inverse, to be

$$Z_0(\tau) = \exp\left(\tfrac{1}{2}\int dp\, J^a_\mu(-p) P_{\mu\nu}(p) J^a_\nu(p)/p^2[1-\exp(-2p^2\tau)]\right). \tag{3.3}$$

The large τ limit of this expression,

$$\exp\left(\tfrac{1}{2}\int dp\, J^a_\mu(-p) P_{\mu\nu}(p) J^a_\nu(p)/p^2\right) \tag{3.4}$$

is the generating functional of the conventional free quantum field theory in the Landau gauge.

By adopting the same procedure as in § 1 it can easily be shown that equation (3.1) follows from the modified Langevin equation

$$dA^a_\mu(p,\tau) = P_{\mu\nu}(p)\left(-\frac{\delta S}{\delta A^a_\nu(-p,\tau)} + \sqrt{2}\, dW^a_\nu(p)\right). \tag{3.5}$$

The generating function obeys

$$\frac{\partial Z}{\partial \tau} = \int dp\, J^a_\mu(p) P_{\mu\nu}(p)\left[-\frac{\delta S}{\delta A^a_\nu}\left(\frac{\delta}{\delta J(p)}\right) + J^a_\nu(-p)\right] Z, \tag{3.6}$$

and the equivalent of equation (2.7) is

$$Z_n(\tau) = \exp\left(\tfrac{1}{2}\int dp\, J^a_\mu(-p) J^a_\mu(p) p^{-2}\right)\int_0^\tau d\tau (W^{(1)}_n + W^{(2)}_n) \tag{3.7}$$

where

$$W_n^{(N)} = \exp\left(-\tfrac{1}{2}\int \mathrm{d}p\, J_\mu^a(-p)J_\mu^a(p)p^{-2}\right)\int \mathrm{d}p\, J_\mu^a(p)P_{\mu\nu}(p)\frac{\delta S^{(N)}}{\delta A_\nu^a}\left(\frac{\delta}{\delta J(p)}\right)Z_{n-N}$$

for $n \geq N$ and zero otherwise.

The evaluation of the right-hand side of equation (3.7) for general τ is tedious. However it is easy to show by induction that the Z_n have a large τ limit independent of τ. Assume that for $n \leq M$

$$Z_n = Q_n[1+\mathrm{O}(\exp(-p^2\tau))]\exp\left(\tfrac{1}{2}\int \mathrm{d}p\, J_\mu^a(-p)P_{\mu\nu}(p)J_\nu^a(p)/p^2\right), \tag{3.8}$$

where the Q_n are polynomials in $P_{\mu\nu}(p)J_\nu(p)$. Using equation (3.7) gives

$$\begin{aligned} Z_{M+1} = {} & \exp\tfrac{1}{2}\int \mathrm{d}p\, J_\mu^a(-p)J_\mu^a(p)/p^2 \\ & \times \int_0^\tau \mathrm{d}\tau\left[[Q'_{M+1}+\mathrm{O}(\exp(-p^2\tau))]\exp\left(-\tfrac{1}{2}\int \mathrm{d}p\, J_\mu^a(-p)p_\mu p_\nu J_\nu^a(p)/p^2\right)\right] \end{aligned} \tag{3.9}$$

where Q'_{M+1} is a polynomial in $P_{\mu\nu}(p)J_\nu(p)$. In equation (3.9) the exponential factor inside the τ integration is independent of τ (use equation (2.8)) and can be taken outside the integral. The integration of the polynomial Q'_{M+1} with respect to τ is easily achieved using equation (2.6), the result being another polynomial. Hence Z_{M+1} has the same form as Z_M. Equation (3.8) is true for $n=0$ and hence by induction for all n.

The large τ solution of equation (3.6) is now simple. Set $\partial Z/\partial\tau$ and note the operator identity

$$\frac{\delta S_I}{\delta A_\mu^a}\left(\frac{\delta}{\delta J(p)}\right) \equiv J_\mu^a(p) - \exp\left[-S_I\left(\frac{\delta}{\delta J(p)}\right)J_\mu^a(p)\exp S_I\left(\frac{\delta}{\delta J(p)}\right)\right] \tag{3.10}$$

which suggests the solution

$$\begin{aligned} Z(\tau=\infty) = {} & N\exp-(gS^{(1)}+g^2S^{(2)})\left(\frac{\delta}{\delta J(p)}\right) \\ & \times\exp\left(\tfrac{1}{2}\int \mathrm{d}p\, J_\mu^a(p)P_{\mu\nu}(p)J_\nu^a(p)/p^2\right), \end{aligned} \tag{3.11}$$

where N is a normalisation constant.

Expression (3.11) for the generating functional differs from the Fadeev-Popov form in not containing the ghost term. The necessity of this term is demonstrated by the following argument. Multiplying equation (3.5) by the momentum vector p shows that

$$\mathrm{d}(p_\mu A_\mu^a(p,\tau)) = 0. \tag{3.12}$$

Condition (3.12) partitions the ensemble of stochastic differential equations according to each member's value of the variable $c^a = p_\mu A_\mu^a$. In order to calculate averages it is therefore necessary to sum over all values of c^a. This can be achieved by functionally integrating with respect to c. The stochastic quantisation hypothesis must be modified to

$$\lim_{\tau\to\infty}\left\langle\int D[c]G[A(x,\tau)]\right\rangle_{\text{noise}} = \langle G[A(x)]\rangle_{\text{FT}}. \tag{3.13}$$

It is more convenient to change the variable in the functional integral from c to the group element U and write

$$\left\langle \int D[c]G \right\rangle_{\text{noise}} = \left\langle \int D[U] \left| \frac{\delta c}{\delta U} \right| G \right\rangle_{\text{noise}}. \tag{3.14}$$

However, for gauge invariant functionals the integrand on the right-hand side of equation (3.14) is independent of the group element U and the functional integral with respect to U can be factored out. The Feynman rules for the stochastic field theory can therefore be derived from the modified generating functional

$$Z_{\text{mod}}(\tau) = \left\langle \left| \frac{\delta c}{\delta U} \right| \exp \int \mathrm{d}x\, A^a_\mu(x, \tau) J^a_\mu(x) \right\rangle_{\text{noise}}. \tag{3.15}$$

This expression can be written as

$$Z_{\text{mod}}(\tau) = \left| \frac{\delta c}{\delta U} \right| \left(\frac{\delta}{\delta J(x)} \right) Z(\tau). \tag{3.16}$$

The functional determinant can be written in terms of the Fadeev–Popov ghost fields. Combining equation (3.11) with equation (3.16) then allows the Feynman rules of the stochastic theory to be read off.

4. Conclusion and outlook

The Ito calculus has been used to study Parisi and Wu's stochastic quantisation procedure for gauge field theory and a differential equation has been derived for the generating functional. In the case of the Abelian theory (free field) this equation is exactly soluble and the stochastic procedure agrees with the functional integral approach in the Landau gauge. For the non-Abelian theory the situation is less clear cut. An attempt was made to solve the generating functional equation by doing perturbation theory about the free field. However this perturbation series approach cannot be justified because the series appears not to converge. A modified stochastic quantisation procedure was proposed which allowed a well defined perturbation theory analysis to be carried out. The modified stochastic procedure was then shown to agree with the Landau gauge functional integral formulation.

An alternative to the generating functional approach adopted in this paper would be to use Ito's theorem to write down the equation of motion of the expectation value of a gauge invariant object. Perturbation theory could then be used to calculate this gauge invariant average in terms of averages of free field gauge invariant quantities. This procedure would reproduce the Landau gauge functional integral formulation minus the ghosts.

The techniques developed in this paper can be applied to scalar field theory (Thomas 1985). An interesting possibility is to derive a differential equation for the generating functional of connected Green functions. This generating functional could then be expanded either in powers of the coupling constant or in powers of Planck's constant. Both types of expansion are under active consideration.

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